The Fifth International Conference on Neural Information Processing
Kitakyushu, Japan October 21-23, 1998

Comparison Among Three Estimators For High Order Statistics

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ABSTRACT

Most of recent and important algorithms in signal processing (for blind identification or separation, *etc*) are based on higher order statistics (HOS). And most of them use a criteria based on the fourth order statistics (moment or cumulant). The problem of using HOS consists on the estimation of the statistics. In the literature, three different estimators of the fourth order cumulant were used. In this paper, we show by an experimental study that the performance of these estimators depend on the nature of the real stochastic signal (stationary or non-stationary). We found that when choosing the estimator, one must take in consideration the signal statistical properties.
KEYWORDS: Higher Order Statistics, Cumulants, Blind Identification, Estimators, Adaptive algorithms.

1. Introduction

In the last two decades, high order stochastic (HOS) methods and theories were one of the most important field in signal processing theory.
The HOS can be considered as an important complement of the classic second order stochastic (SOS) methods (power, variance, covariance and spectra) to solve many recent and important telecommunication problems[1], as blind identification or equalization, blind separation of sources and time delay estimation [2, 3, 4, 5, 6]. Most of these HOS algorithms are based on the fourth order statistics.
In this paper, we focus an the estimation problem of the second and fourth order statistics, which are the mostly used ones.

By definition [7], the rth order moment μ_r of a stochastic signal X is:

$$\mu_r = E[X^r] \tag{1}$$

where E is the mathematical expectation. The rth order cumulant of X can be calculated from its moments, by using the Leonov-Shiryayev formula[1] [9], [10]:

$$\begin{aligned} Cum_r[X] &= Cum[X, \ldots, X] \\ &= \sum (-1)^{k-1}(k-1)! E[X^{v_1}]E[X^{v_2}] \ldots E[X^{v_p}] \end{aligned} \tag{2}$$

By using this relationship, we calculate the 4th order cumulant of X :

$$\begin{aligned} Cum_4[X] \quad = \quad & E[X^4] - 4E[X]E[X^3] - 3E^2[X^2] \\ & +12E^2[X]E[X^2] - 6E^4[X]. \end{aligned} \tag{3}$$

For a zero-mean stochastic signal [2], the second order cumulant is equal to its second order moment and its 4th order cumulant becomes:

$$Cum_4[X] = E[X^4] - 3E^2[X^2]. \tag{4}$$

2. Classical estimator

Let X to be a zero mean stochastic signal where x_i is an event (or a signal sample) of X ($1 < i < N$). The classic estimator of the rth order moment of X is given by:

$$\widehat{\mu_r} = \frac{1}{N} \sum_{i=1}^{N} x_i^r. \tag{5}$$

It is easy to verify that (5) is an unbiased estimator of the rth order moment of X (i.e $E[\widehat{\mu_r}] = \mu_r$).
To estimate the 4th order cumulant of X, we can derive an estimator from the Leonov-Shiryayev formula (4):

$$\widehat{Cum}_4[X] = \widehat{\mu_4} - 3\widehat{\mu_2}^2, \tag{6}$$

It is proved [11, 10] that the estimator (6) is a biased estimator and the estimation error decreases proportional to $\frac{1}{N}$. In fact, by using (5) and (6) we can write:

$$\begin{aligned} \widehat{Cum}_4[X] &= \widehat{\mu_4} - 3\widehat{\mu_2}^2, \\ &= \frac{1}{N} \sum_{i=1}^{N} x_i^4 - \frac{3}{N^2} \sum_{i,j=1}^{N} x_i^2 x_j^2, \end{aligned} \tag{7}$$

[1] The original formula shows the relationship among the cumulant of r stochastic signals X_i ($i = 1, \ldots, r$) and theirs moments of order $p, p \le r$:

$$Cum[X_1, \ldots, X_r] = \sum (-1)^{k-1}(k-1)! E[\prod_{i \in v_2} X_i] E[\prod_{j \in v_2} X_j] \ldots E[\prod_{k \in v_p} X_k]$$

where the addition operation is over all the set of v_i ($1 \le i \le p \le r$) and v_i compose a partition [8] of $\{1, \ldots, r\}$.

[2] In many applications, the stochastic signal X is zero mean signal.

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And the estimator expectation becomes:

$$E[\widehat{Cum}_4[X]] = \mu_4 - \frac{3}{N}(\mu_4 + (N-1)\mu_2^2)$$
$$= Cum_4[X] - \frac{3}{N}(Cum_4[X] + 2\mu_2^2) \quad (8)$$

It is easy to prove that:

$$\widehat{Cum}_4[X] = \frac{N+2}{N(N-1)} \sum_{i=1}^{N} x_i^4 - \frac{3}{N(N-1)} \sum_{i,j=1}^{N} x_i^2 x_j^2, \quad (9)$$

is an unbiased estimator for the fourth order cumulant of X. The estimation error between the two estimators (7) and (9) depends at first on the samples number N and on the statistics of X. By experimental study, we found that when $N > 100$, the estimation error will become very small (see figure 1).

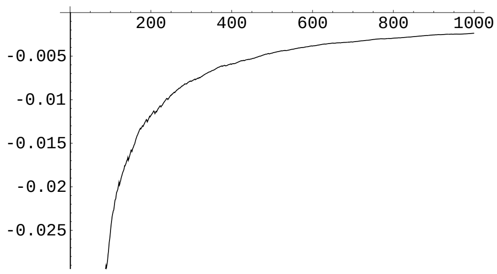

Figure 1. The estimation error for the classical estimator of the fourth order cumulant.

The estimator (5) is not an adaptive one, but it is easy to force it to be an adaptive one:

$$\widehat{\mu_r}\{k\} = \frac{1}{k}\sum_{i=1}^{k} x_i^r = \frac{(k-1)\widehat{\mu_r}\{k-1\} + x_k^r}{k} \quad (10)$$

where $\widehat{\mu_r}\{k\}$ is the estimator of the rth order moment at the kth iteration. Finally, by using (9) and (10) we can say that the classical adaptive estimator for the 4th order statistics is:

$$\widehat{\mu_2}\{k\} = \frac{(k-1)\widehat{\mu_2}\{k-1\} + x_k^2}{k} \quad (1 \le k \le N)$$
$$\widehat{\mu_4}\{k\} = \frac{(k-1)\widehat{\mu_4}\{k-1\} + x_k^4}{k} \quad (1 \le k \le N)$$
$$\widehat{Cum}_4[X]\{k\} = \frac{k+2}{k-1}\widehat{\mu_4}\{k\} - \frac{3k}{k-1}(\widehat{\mu_2}\{k\})^2 \quad (11)$$
$$(1 < k \le N)$$

This algorithm is very simple and its convergence is very fast, but it may have difficulty to converge when the signal is non-stationary (see section 4).

3. Low-pass estimators

The classical estimator is not the only estimator used in the literature to estimate the fourth order statistics. Many authors use an adaptive estimators, these estimators are named low-pass estimators:

$$\widehat{\mu_2}\{k\} = (1-\alpha)\widehat{\mu_2}\{k-1\} + \alpha x_k^2$$
$$\widehat{\mu_4}\{k\} = (1-\gamma)\widehat{\mu_4}\{k-1\} + \gamma x_k^4$$
$$\widehat{Cum}_4[X]\{k\} = \widehat{\mu_4}\{k\} - 3(\widehat{\mu_2}\{k\})^2 \quad (12)$$

where $(1 \le k \le N)$. The greatest advantage of this estimator consists on that it is very simple and it can be used even when the signal is non-stationary (see next section). The major problems of this estimator consist on the choice of the values of α and γ, and the variance of the estimation error depends on the signal power (see also the next section).
To estimate the fourth order cumulant, Amblard and Brossier [12] proposed another adaptive low-pass algorithm. Their algorithm avoids the estimation of the fourth-order moment, in fact:

$$\widehat{\mu_2}\{k\} = (1-\alpha)\widehat{\mu_2}\{k-1\} + \alpha x_k^2$$
$$\widehat{Cum}_4[X]\{k\} = \widehat{Cum}_4[X]\{k-1\}$$
$$+\gamma H_k(\widehat{Cum}_4[X]\{k-1\}) \quad (13)$$

with:

$$H_k(\widehat{Cum}_4[X]\{k-1\}) =$$
$$x_k^4 - 3x_k^2\widehat{\mu_2}\{k-1\} - \widehat{Cum}_4[X]\{k-1\}. \quad (14)$$

The quantity $x_k^4 - 3x_k^2\widehat{\mu_2}\{k-1\}$ is a kind of instantaneous estimate of the cumulant, so that $H_k(\widehat{Cum}_4[X]\{k-1\})$ measures the distance between the previous value $\widehat{Cum}_4[X]\{k-1\}$ and the instantaneous estimate. This algorithm is simple and asymptotically unbiased, but it is relatively slow to converge. Amblard *et al.* [13] apply their algorithm into different application: transient detection, blind deconvolution and timing recovery in communication.

4. Experimental results

In this section, we present some experimental results to compare the three estimators (11), (12) and (13). And we estimate the second and the fourth order statistics using these different algorithms. The estimation results (of the three estimators) and the theoretical values (15) are plotted by using different color lines as follows:

- The theoretical values are plotted using a black line.
- The red line corresponds to the classical estimator.
- The blue line corresponds to the low-pass estimator.
- The green line corresponds to Amblard'estimator (just for the fourth order cumulant).

At first, let us consider the case when the signal X is a white, zero-mean and stationary signal with a uniform probability density function (pdf). The fourth order statistics of this signal are equal to:

$$E[X^2] = \frac{A^2}{3}$$

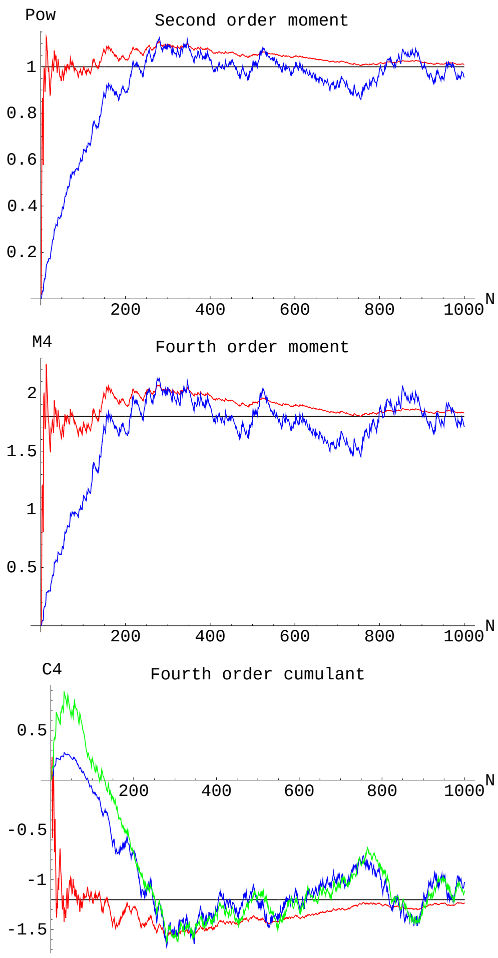

Figure 2. White, zero-mean and stationary signal: $\alpha = 0.01$, $\gamma = 0.03$.

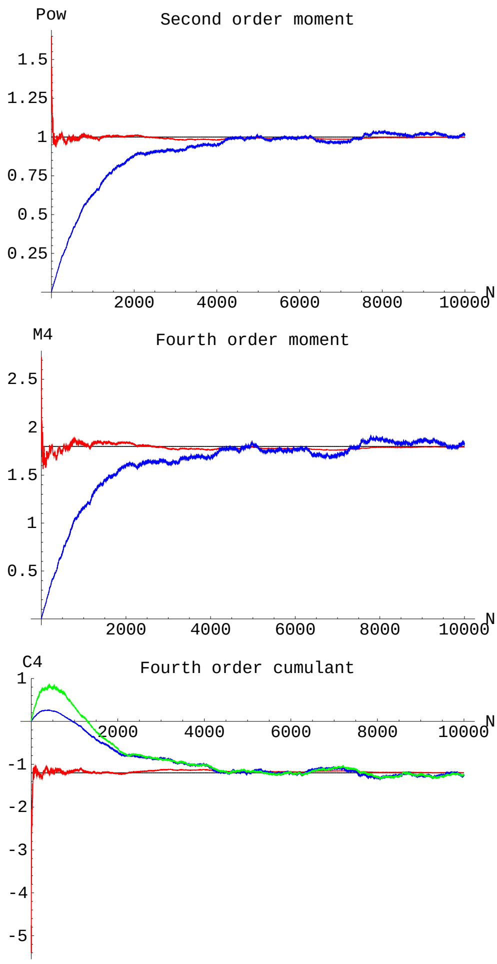

Figure 3. White, zero-mean and stationary signal: $\alpha = 0.001$, $\gamma = 0.003$.

$$
\begin{aligned}
E[X^4] &= \frac{A^4}{5} \\
Cum_4[X] &= \frac{-2A^4}{15}
\end{aligned}
\tag{15}
$$

Where A is the maximum amplitude of X. For stationary signal, the classical estimator (11) converge very fast and it seems that it gives better performances than the other estimators (see figure 2).

As we said in the previous section, the performances of the low-pass estimator (12) and the convergence speed of Amblard-Brossier estimator depend on the value of two parameters α and γ. Thus, when we decrease (resp. increase) the values of α and γ than the convergence speed will decrease (resp. decrease) but the variance of the estimation error will increase (resp. decrease). For the same signal and for different values of α and γ, we can observe in figure 3 the variation of the estimator performances.

Unfortunately, even though the classical estimator was the good estimator for stationary signal, becomes useless for a non-stationary signal. For example, in figure 4, we estimate the second and the fouth order statistics over a 30000 samples of an iid zero-mean non-stationary signal, this signal composes of four parts: uniform pdf (5000 samples), Gaussian pdf (3000), uniform pdf (10000) and the last part has a Gaussian pdf (12000). We can observe in the same figure 4 that the estimation error variance of the low-pass estimator depends on the signal variance, but it converged faster than the Amblard-Brossier estimator.

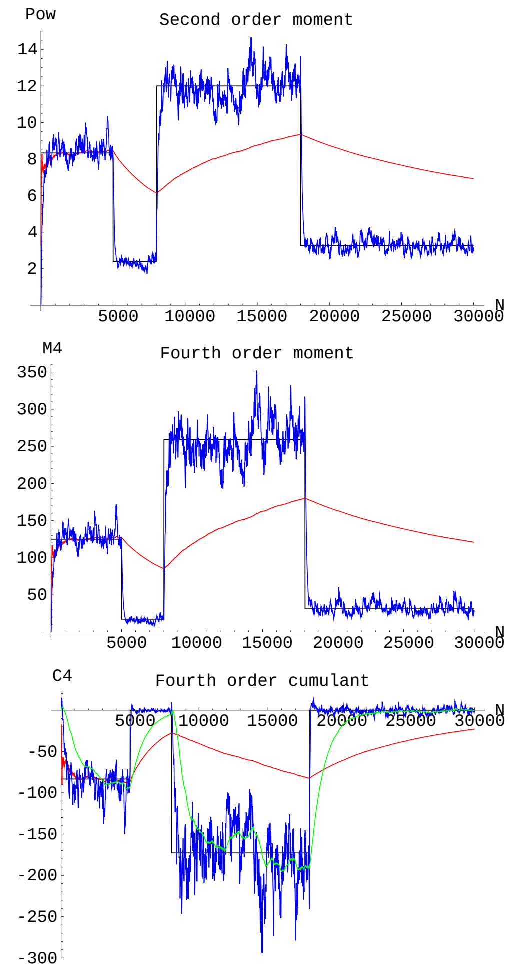

Figure 4. Non-stationary signal: $\alpha = 0.01$, $\gamma = 0.001$.

5. conclusion

In this paper, we present an experimental study to compare three estimators (classical estimator, low-pass estimator and Amblard-Brossier estimator) of higher order statistics of real signal. The first two estimators can be used to estimate the moment of any order, and they are unbiased estimators for the moments. In general case, the cumulant estimators of any order (greater than 3) are biased (except the fourth order cumulant estimator in (11) and the Amblard-Brossier estimator which is asymptotically unbiased estimator also for the fourth order cumulant).

For stationary signals, the classical estimator converge very fast and gives better performance than the two other estimators. And the performance of the low-pass estimator and Amblard-Brossier estimator depend on the choice of two parameters α and γ.

For non-stationary signals, the classical estimator has a real problem to converge. Thus, in this case it will be better to use the low-pass or the Amblard-Brossier estimator. To estimate the fourth order cumulant of non-stationary signals, the performance of Amblard-Brossier estimator is better in general than the low pass estimator because the estimation error variance of the low-pass estimator depends on the signal variance.

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