A Simple Cost Function For Instantaneous and Convolutive Sources Separation

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RÉSUMÉ

Dans le problème de séparation aveugle de sources, les solutions sont généralement basées sur une fonction de coût: fonction de contraste [11], [3], ou polynôme dépendant de statistiques d'ordre supérieur [13], [5], et [7]. Dans ce papier, nous montrons qu'un critère simple, utilisant uniquement des cumulants croisés Cum_{22}, permet de séparer N sources à partir de N mélanges instantanés ou convolutifs, pourvu que les sources aient des kurtosis de même signe.

ABSTRACT

In general case, the solution of the blind separation of sources is based on a cost function, which may be a contrast function [11], [3], or a polynomial function of higher order statistics [13], [5], and [7]. In this paper, we propose a simpler cost function, based on 4th-order cross-cumulants can be used for the separation of N sources from N mixtures, instantaneous as well as convolutive, provided than sources have the same sign of kurtosis.

1 Introduction

First solutions of separation of sources, proposed in [5], was based on cancellation of higher order output cross-moments $Ef(s_i)g(s_j)$. To avoid some limitations, other cost functions: contrast function [3] [11] or polynomial function of 4th-order cumulants [2], have been proposed.

In [9], we proved the efficacy of a simple cost function for instantaneous mixtures in the case of 2 sources and 2 sensors. In this paper, we generalize the result for instantaneous as well convolutive mixtures, with N sources and N sensors. We also show the cost function is efficient if mixtures are corrupted with additive Gaussian noise.

2 Equation model

Let us consider N zero-mean unknown sources $s_i(t)$, assumed statistically independent. and N sensors, outputs of which are unknown linear convolutive mixtures $x_i(t)$ (in the most general case) of the sources. Let us denote $M(z)$ the polynomial mixture matrix: $M(z) = \sum_{p=0}^{N_m} M(p)z^{-p}$, where $M(p)$ is NxN matrix, and N_m is the order of the filter. Denoting $\vec{X}(n)$ the mixture vector, and $\vec{S}(n)$ the source vector at any time n, we have: $\vec{X}(n) = \sum_{p=0}^{N_m} M(p)\vec{S}(n-p)$.

The separation is achieved by estimating a NxN polynomial matrix $W(z)$ satisfying $W(z)M(z) = PD(z)$, where P and D are any permutation and polynomial diagonal matrix, respectively. The global matrix $W(z)M(z)$ will be denoted $H(z) = (h_{ij}(z)) = \sum_0^{N_h} H(p)z^{-p}$, where N_h is the filter order:

$$\vec{Y}(n) = \sum_{k=0}^{N_h} H(p)\vec{S}(n-p), \text{ and } \vec{Y}(z) = H(z)\vec{S}(z). \quad (1)$$

In tensorial notation [10], the last equation can be written:

$$\begin{aligned} Y^1(n) &= \sum_k H_1^1(p) \bullet S^1(n-p), \quad (2) \\ Y^1(z) &= H_1^1(z) \bullet S^1(z), \end{aligned}$$

where $Y^1(z)$ and $S^1(z)$ are two contravariant tensors, H_1^1 is an one time covariant and one time contravariant tensor, and finally $\bullet$ is the contraction process[1].

[1]The contraction is equivalent of the matrix product. So the general coefficient of $A^1 = B_1^1 \bullet C^1$ is $a^i = \sum_j b_j^i c^j$

3 Instantaneous mixture

3.1 Mixtures without noise

For instantaneous mixture model, issues of matrices (M, W and H) are scalar. Assuming the observations are not corrupted by noise, one has $\vec{X} = M\vec{S}$, and one is looking for a matrix W such that $\vec{Y} = W\vec{X} = H\vec{S}$, where $H = WM = DP$.

Denoting $Cum_{22}(y_i, y_j) = Cum(y_i, y_i, y_j, y_j)$, and using multi-linearity properties of cumulants (see [12]) and source independence, we have:

$$Cum_{22}(y_m, y_n) = \sum_i h_{mi}^2 h_{ni}^2 \beta_i, \qquad (3)$$

where $\beta_i = Cum(s_i, s_i, s_i, s_i)$. Let us consider the functions $J_s = \sum_{m<n} Cum_{22}^2(y_m, y_n)$, and $J_a = \sum_{m<n} |Cum_{22}(y_m, y_n)|$. Assuming sources have the same sign of kurtosis, $J_s = 0$ or $J_a = 0 \iff h_{mi}h_{ni} = 0$, $\forall i$, and $m \neq n$. This hypothesis on kurtosis is realistic in many situations, especially for telecommunication signals and has been used by several authors [9], [11], [8].

So each column vector of global matrix H has at most one coefficient not equal to zero (suppose that $h_{mi} \neq 0$. Then, equation (3) will be satisfied if only if $h_{(m+i)a} = 0$, $\forall 1 \leq i \leq (N-m)$). The global matrix H cannot have 2 coefficients different from zero in the same column, but it could have 2 coefficients different from zero in the same row. We will prove in the following that this case may be avoided using a simple constraint on W.

We constrain matrix W such that $w_{ii} = 1$. Then global matrix is a permutation matrix. Let us consider the mixture matrix M like a set of column vectors: $M = (\vec{M}_1, \ldots, \vec{M}_N)$. And let us denote $\vec{W}_i^T$, the i-th row of matrix W. The assumption $w_{ii} = 1$ implies that $\vec{W}_i^T \neq \vec{0}$. If $\vec{H}_i^T$, the i-th line of H, is equal to zero, then all the coefficients of this line are zero:

$$h_{ij} = \vec{W}_i^T \vec{M}_j = 0. \qquad (4)$$

Mixture matrix M being a regular matrix, vectors $\vec{M}_i$ are linearly independent, it is impossible to satisfy the relation (4), because it does not exist a real vector in $\Re^N$ not equal to zero and orthogonal to N linearly independent vector. With the constraint $w_{ii} = 1$, we then deduce each row of the global matrix H is not equal to zero. There is at least one issue not equal to zero on each row of H. Moreover, we already know each column of H has at most one non zero issue, we deduce there is one and only one non zero issue per row and per column. The matrix H then satisfies $H = PD$.

3.2 Mixtures with Gaussian noise

Let us denote N^1 a noise tensor, component of which are assumed Gaussian, and independent of the sources. Then, the mixture tensor is:

$$X^1 = M_1^1 \bullet S^1 + N^1, \qquad (5)$$

and the estimated sources are:

$$Y^1 = H_1^1 \bullet X^1 + W_1^1 \bullet N^1. \qquad (6)$$

In the following, we will denote $\bigotimes$ the tensorial product[2], and we will use Einstein notation[3] (see [1]). Let us now consider the 4-th cumulant Cum_{22}. In general case, it is a 4-th order tensor, two times covariant and two times contravariant [10], and will be denoted Cum_2^2. Using multi-linear property of cumulant, source independence, and independence between signals and noises, we may expand $Cum_{Y_2^2} = Cum(Y_1, Y^1, Y_1, Y^1)$ from relation (6) ([10], [4]):

$$\begin{aligned} Cum_{Y_2^2} &= H_1^1 \bigotimes H_1^{1^T} \bigotimes H_1^1 \bigotimes H_1^{1^T} \bullet Cum_{S_2^2} \\ &+ W_1^1 \bigotimes W_1^{1^T} \bigotimes W_1^1 \bigotimes W_1^{1^T} \bullet Cum_{N_2^2}. \end{aligned} \qquad (7)$$

Noise being a Gaussian vector, then $Cum_{N_2^2}$ equals to zero. Moreover, because of source independence, $Cum_{S_2^2}$ reduces to a diagonal tensor. Denoting δ_{mn}^{op} the generalized Kronecker symbol, the general term of $Cum_{Y_2^2}$ becomes:

$$\begin{aligned} Cum_{Y_{2jl}^{2ik}} &= H_m^i H_j^{nT} H_o^k H_l^{pT} Cum_{S_{2mn}^{2mn}} \delta_{mn}^{op} \\ &= H_m^i H_m^j H_m^k H_m^l Cum_{S_{2mm}^{2mm}}. \end{aligned} \qquad (8)$$

From (8), we deduce $Cum_{Y_{2ik}^{2ik}} = (H_m^i)^2 (H_m^k)^2 \beta_m$. This relation is similar to (3). Assuming the constraint $w_{ii} = 1$ on the matrix W, and using results of the last subsection, we know the global matrix H is equal to $H = PD$. This means that cancelling or minimizing J_s or J_a leads to $W = PDM^{-1}$ and:

$$\vec{Y} = PD\vec{S} + PDM^{-1}\vec{N}, \qquad (9)$$

where D and P are diagonal and permutation matrices, respectivily. As a conclusion, J_s and J_a are efficient cost functions, even with additive Gaussian noise.

[2] The tensorial product of two tensors, B_1^1 and C_2^1, is the tensor $A_3^2 = B_1^1 \bigotimes C_2^1$, general term of which is $a_{ijk}^{lm} = b_i^l c_{jk}^m$.

[3] Example: the sum $a_l = \sum_{ij} b_{li} c_{ij} d_j$, will be reduced using Einstein notations to $a_l = b_{li} c_{ij} d_j$.

However, let us remark that the estimated sources are not perfectly separated, but are corrupted by the noise, as it is shown by equation (9).

4 Convolutive mixture

In the case of convolutive mixture, using Einstein notation, equation (2) can be rewritten:

$$Y_1(n) = H_1^1(p) \bullet S_1(n-p). \qquad (10)$$

Considering tensors $Y_1(n)$ and $Y^1(n)$ at different time, (7) can be generalized:

$$\begin{aligned}&Cum(Y_1(p_1), Y^1(p_2), Y_1(p_3), Y^1(p_4)) =\\ &\quad H_1^1(q_1) \bigotimes H_1^{1T}(q_2) \bigotimes H_1^1(q_3) \bigotimes H_1^{1T}(q_4)\\ &\quad \bullet Cum(S_1(p_1-q_1), S^1(p_2-q_2),\\ &\quad S_1(p_3-q_3), S^1(p_4-q_4)).\end{aligned}$$

The general term of this tensor is:

$$\begin{aligned}&Cum_{Y}{}_{ij}^{kl}(p_1,p_2,p_3,p_4) =\\ &\quad H_i^a(q_1)H_b^{jT}(q_2)H_k^c(q_3)H_d^{lT}(q_4)\\ &\quad Cum(s_a(p_1-q_1), s_b(p_2-q_2),\\ &\quad s_c(p_3-q_3), s_d(p_4-q_4)),\end{aligned} \qquad (11)$$

where $H_i^a(q_1)$ (respectively $H_i^{aT}(q_1)$ is the coefficient $h_{ia}(q_1)$ (respectively $h_{ja}(q_1)$) of the matrix $H(q_1)$ (see section 2).

Using source independence assumption, we have:

$$\begin{aligned}&Cum(s_a(p_1-q_1), s_b(p_2-q_2),\\ &\quad s_c(p_3-q_3), s_d(p_4-q_4)) =\\ &\quad \delta_{ab}^{cd} Cum(s_a(p_1-q_1), s_b(p_2-q_2),\\ &\quad s_c(p_3-q_3), s_d(p_4-q_4)).\end{aligned}$$

Then, relation (11) becomes:

$$\begin{aligned}&Cum_{Y}{}_{ij}^{kl}(p_1,p_1,p_3,p_4) =\\ &\quad H_i^a(q_1)H_a^{jT}(q_2)H_k^a(q_3)H_a^{lT}(q_4)\\ &\quad Cum_{s_a}(n-q_1-p_1, n-q_2-p_2,\\ &\quad n-q_3-p_3, n-q_4-p_4).\end{aligned}$$

Finally, using $H_i^a(q_1) = h_{ia}(q_1)$, the above relation can be written:

$$\begin{aligned}&Cum_{Y}{}_{ij}^{kl}(p_1,p_2,p_3,p_4) =\\ &\quad h_{ia}(q_1)h_{ja}(q_2)h_{ka}(q_3)h_{la}(q_4)\\ &\quad Cum_{s_a}(p_1-q_1, p_2-q_2, p_3-q_3, p_4-q_4).\end{aligned} \qquad (12)$$

Assuming that the sources s_a are iid and stationary signals, the cumulant simplifies to:

$$\begin{aligned}&Cum_{s_a}(p_1-q_1, p_2-q_2, p_3-q_3, p_4-q_4) =\\ &\quad \delta_{(q_1+p_1)(q_2+p_2)}^{(q_3+p_3)(q_4+p_4)} Cum_{s_a}(p,p,p,p).\end{aligned}$$

Denoting $Cum_{s_a}(p,p,p,p)$ by β_a, we deduce from equation (12):

$$\begin{aligned}&Cum_{Y}{}_{ij}^{kl}(p_1,p_2,p_3,p_4) = 0 \Rightarrow\\ &\quad h_{ia}(q_1)h_{ia}(q_1+p_1-p_2)\\ &\quad h_{ja}(q_1+p_1-p_3)h_{ja}(q_1+p_1-p_4)\beta_a = 0.\end{aligned} \qquad (13)$$

Filters $h_{ia}(z)$ are M-order causal filters, consequently $h_{ia}(p) = 0$ for $p < 0$ and $p > M$. Then, in equation (13), the q_1 parameter is limited by $\max(0, p_2-p_1, p_3-p_1, p_4-p_1) \le q_1 \le \min(M, M+p_2-p_1, M+p_3-p_1, M+p_4-p_1)$.

Finally, choosing $p_1 = p_2 = 0$ and $p_3 = p_4 = q_1 - q_2$, where q_1 and q_2 are any integer, equation (13) is:

$$h_{ia}^2(q_1)h_{ja}^2(q_2)\beta_a = 0. \qquad (14)$$

If the sources have the same sign of kurtosis, the last equation give us that $h_{ia}(q_1)h_{ja}(q_2) = 0, \forall a, q_1, q_2$, and $i \neq j$. This hypothesis make that, when $q_1 = q_2$ then $H(q_1) = PD$, where P is a permutation matrix, and D is a diagonal matrix, see [9]). The result may be generalized for $q_1 \neq q_2$, and $H(q_2) = PD'$ is then found, where D' is another diagonal matrix and P is the same permutation matrix. Finally, the filter will be $H(z) = P\sum_i D(i)z^{-i}$. So the separation is achieved up to a permutation and a polynomial diagonal matrix.

4.1 Using z-domain cumulants

The same result can be proved using the generalisation of cumulant definition in the z domain. For example, the fourth order cumulant of $S(z)$ will be defined as $Cum_4(S(z)) = ES^4(z) - 3E^2S^2(z)$. Applying this definition on z-transform of (10), we may derive the condition: $Cum_{ij}^{ij}Y(z) = 0 \Rightarrow h_{ia}^2(z)h_{ja}^2(z)\beta_a(z) = 0$. Assuming that $M(z)$ is a column-reduced matrix (see [6]) and $w_{ii}(z) = 1$, we can prove that $H(z)$ is a polynomial diagonal matrix up to a permutation.

5 Conclusion

In this paper, we propose a cost function which is the sum of squared (or absolute value) cross-cumulants:

$J_s = \sum_{i<j} Cum^2(y_i(n), y_i, y_j, y_j)$ (or $J_a = \sum \mid Cum(y_i, y_i, y_j, y_j) \mid$).

- Cum_{22} is a sufficient cost function for instantaneous mixture if sources have the same sign of kurtosis. The result still holds if the mixture is corrupted with additive Gaussian noise.

- If the sources are iid and have the same sign of kurtosis, the result can be generalized to convolutive mixture using the simple cost $J(n) = \sum_{i,j,p} Cum^2(y_i(n), y_i(n), y_j(n-p), y_j(n-p))$. In that case, tensorial and Einstein notations are very convenient to simplify equations.

Experimental results in the case of two sensors and two sources, were developped in [9] for instantaneous mixtures and in [13] for convolutive mixtures. Unfortunately, cumulants Cum_{22} are flat around solutions and do not lead to very precise results. However, the theoretical results obtained in this paper could be used to verify the relevance of solutions provided by a simpler criterion.

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